Extending finite group actions on surfaces over S^3

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Abstract

Let OE_g (resp. CE_g and AE_g) and resp. OE_g^o be the maximum order of finite (resp. cyclic and abelian) groups G acting on the closed orientable surfaces Σ_g which extend over (S^3, Σ_g) among all embeddings $\Sigma_g \to S^3$ and resp. unknotted embeddings $\Sigma_g \to S^3$.

It is known that $OE_g^o \le 12(g-1)$, and we show that $12(g-1)$ is reached for an unknotted embedding $\Sigma_g \to S^3$ if and only if $g = 2$, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241, 601. Moreover AE_g is $2g+2$; and CE_g is $2g+2$ for even g, and $2g-2$ for odd g.

Efforts are made to see intuitively how these maximal symmetries are embedded into the symmetries of the 3-sphere[1].

Contents

1 Introduction

In this article, we use Σ_g to denote the orientable closed surface of genus $g > 1$, and use V_g to denote the handlebody of genus $g > 1$. All group actions will be faithful and orientation-preserving (on both surfaces and 3-manifolds).

Let O_g (resp. C_g and A_g) be the maximum order of all finite (resp. cyclic and abelian) groups G which can act on Σ_g. A classical result of Hurwitz

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states that O_g is at most $84(g-1)$, proved by applying the Riemann-Hurwitz formula. However for each fixed g, O_g is hard to determine in general, see [Ac], [Ma] for some partial results. On the other hand C_g is $4g+2$, see [St], also [Ha], and [Wa] for more direct constructions; and A_g is $4(g+1)$, see [Ma].

Let OH_g (resp. CH_g and AH_g) be the maximum order of all finite (resp. cyclic and abelian) groups G which can acts on the handlebody V_g. By definition we have that $O_g \geq OH_g, C_g \geq CH_g$ and $A_g \geq AH_g$. It is a result due to Zimmermann [Zi1] that $OH_g \leq 12(g-1)$, see also [Zi2], [MMZ], [MZ]. A handlebody orbifold theory was derived in [MMZ], see also [Zi3] for a more geometric approach, and then proved that CH_g is $2g+2$ when g is even, and $2g$ when g is odd [MMZ]. Moreover OH_g is bounded below by $4(g+1)$, and OH_g is either $12(g-1)$ or $8(g-1)$ if g is odd, and each of these is achieved by infinitely many odd g [MZ].

In the present article, we consider the following extension problems: Suppose a finite group G acts on the surface Σ_g. If there is an embedding $e : \Sigma_g \subset S^3$ such that G can act on the pair (S^3, Σ_g) and the restriction to Σ_g is the given G action on Σ_g, we call the action of G on Σ_g extendable (over S^3 with respect to e).

Call an embedding $e_o : \Sigma_g \to S^3$ unknotted if each component of $S^3 \backslash \Sigma_g$ is a handlebody. So each extendable G action w.r.t. an unknotted embedding e_o provides a G-invariant Heegaard splitting of S^3. Similarly we define an action of G on V_g to be extendable, and an embedding $e_o : V_g \to S^3$ to be unknotted if the complement $S^3 \setminus V_g$ is also a handlebody. For each g, an unknotted embedding is unique up to isotopy of S^3 and automorphisms on Σ_g (resp. V_g).

In such extension problems, we first study the maximum orders in the present paper. Let OE_g (resp. CE_g and AE_g) be the maximum order of all extendable finite (resp. cyclic and abelian) groups G acting on Σ_g. It is not obvious that $OH_g \geq OE_g, CH_g \geq CE_g$ and $AH_g \geq AE_g$. So finer notions may be useful at the moment: Let OE_g^o (resp. CE_g^o and AE_g^o) be the maximum order of a finite (resp. cyclic and abelian) group G acting on Σ_g which extends over S^3 w.r.t. an unknotted embedding. Then

(1) $OE_g \geq OE_g^o, CE_g \geq CE_g^o$ and $AE_g \geq AE_g^o$.

(2) $OH_g \geq OE_g^o, CH_g \geq CE_g^o$ and $AH_g \geq AE_g^o$.

Now we are going to describe the results and the organization of the paper.

Even to determine OE_g and OE_g^o are harder, some discussions are made in Section 2. It is clear $OE_g^o \leq 12(g-1)$. We show that there are only finitely many g such that $OE_g^o = 12(g-1)$, and indeed we list all such g (Theorem 2.1, see also the Examples in Section 4). It is derived that for each g, $4(g+1)$ is a lower bound for OE_g^o (Example 4.3), and for each $g = n^2$ a lower bound of OE_g^o is $4(n+1)^2$ which is larger than $4(g+1)$ (Example 4.4).

In Section 3, we discuss the abelian case and the cyclic case which are easier. By applying the handlebody orbifold theory, we will first derive the needed information about orders of large abelian and cyclic group action

on V_g (Theorem 3.1 and Theorem 3.2). Then we show that AE_g is $2g+2$ (Theorem 3.3), and CE_g is $2g+2$ for even g, and $2g-2$ for odd g (Theorem 3.5). All these maximum order group actions are realized by unknotted embedding (Examples 4.1 and 4.2), hence $CE_g^o = CE_g$ and $AE_g^o = AE_g$.

Question 1. *If an embedding $\Sigma_g \to S^3$ realizes OE_g, should the embedding be unknotted? Weakly does $OE_g = OE_g^o$ for each $g > 1$?*

The existence of extendable group actions on surfaces with large symmetry presented in Sections 2 and 3 are mostly derived from the orbifold theory. On the other hand in the process of this work, most large symmetries in Sections 2 and 3 are first constructed in a more direct and intuitive way without using orbifold theory. Section 4 presents those constructions which show us how those symmetries on surfaces stay in the symmetries on 3-sphere. A reason of doing so is given in the next paragraph.

We end the introduction by the some motivations of our study: Surfaces (as well as handlebodies) are very familiar subjects to us, mostly because we can see them staying in our 3-space in various manners. The symmetries of the surfaces have been studied for a long time, and it will be natural to wonder when these symmetries can be embedded into the symmetries of our 3-space (3-sphere). Another inspiring fact is a related problem on extending surface automorphisms over 4-space which had been addressed 30 years ago [Mo], and for recent developments see [Hi], [DLWY] and [LNSW].

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2 Maximum order of extendable group action

From now on all groups in this paper will be finite.

A simple picture we should keep in mind is the following: Suppose that the action of G on Σ_g is extendable w.r.t. some embedding $\Sigma \subset S^3$. Let $\Gamma = \{x \in S^3 | \exists g \in G, s.t. gx = x\}$; then Γ is a graph, possibly disconnected, and S^3/G is a 3-orbifold whose singular set Γ/G is also a graph. Each edge of Γ/G can be labeled by an integer > 0 which corresponds to the singular index of it. Also, Σ_g/G is a 2-orbifold with singular set $\Sigma_g/G \cap \Gamma/G$, which are isolated points.

We recall the handlebody orbifold theory from [MMZ], [Zi3].

Let G be a finite group acting on a handlebody of genus g. Associated to this action there is a handlebody orbifold V_g/G, a finite graph of finite groups $(\Gamma, \mathcal{G})$ and a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \longrightarrow G$ whose kernel is isomorphic to a free group of rank g; in particular, ϕ is injective on the finite vertex groups of $(\Gamma, \mathcal{G})$. Here $\pi_1(\Gamma, \mathcal{G})$ denotes the fundamental group of the graph of groups or of the corresponding handlebody orbifold: this is the iterated free product with amalgamation and HNN-extension over the vertex groups, amalgamated over the edge groups of a maximal tree, with

the HNN-generators corresponding to the edges in the complement of the chosen maximal tree. We denote by

$$\chi = \chi(\Gamma, \mathcal{G}) = \sum 1/|G_v| - \sum 1/|G_e| \qquad (2.1)$$

the Euler characteristic of the graph of groups $(\Gamma, \mathcal{G})$ (the sum is taken over all vertex groups G_v resp. edge groups G_e of $(\Gamma, \mathcal{G})$); then

$$g - 1 = |G|(-\chi) \qquad (2.2).$$

The vertex groups G_v belongs to one of the following five classes which correspond to the five types of finite subgroups of the orthogonal group $SO(3)$ given in Figure 1. The edge groups G_e are cyclic groups which are either trivial or maximally cyclic in the adjacent vertex groups. We can also assume that the edge group of an edge which is not a loop (not closed) does not coincide with one of the two vertex groups of the edge.

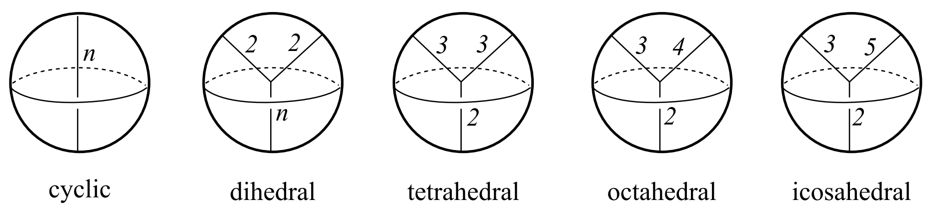

Figure 1

Conversely, to each such finite graph of finite groups associated to a handlebody orbifold and surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \to G$, there is a corresponding action of G on a handlebody V_g of genus g.

Theorem 2.1. *Suppose the G-action on Σ_g is extendable over S^3 w.r.t. the unknotted embedding e_o and the order of G is $12(g-1)$. Then g is as follows: g = 2, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241, 601.*

Proof. The values of g can be obtained as follows.

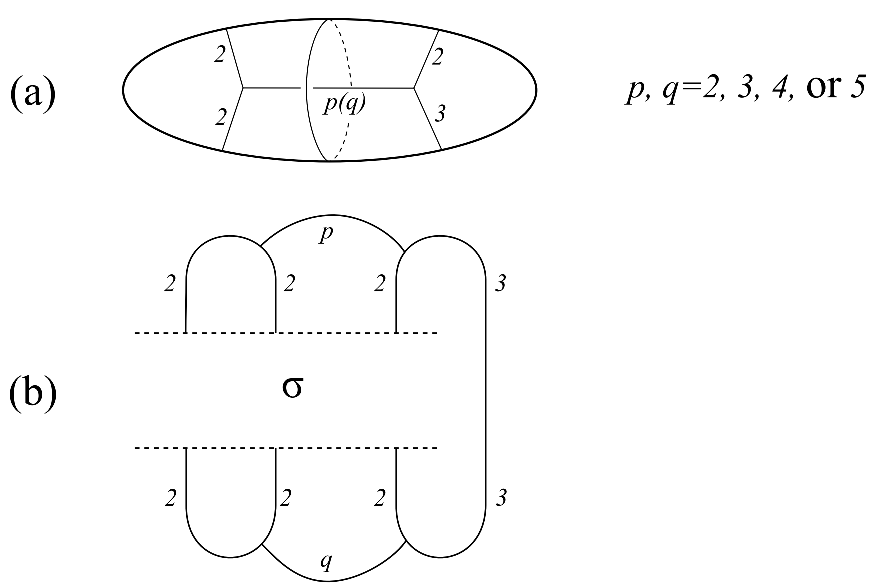

Figure 2

The unknotted embedding $\Sigma_g \to S^3$ provides a Heegaard splitting $S^3 = H_1 \cup_{\Sigma_g} H_2$ such that each handlebody H_i is invariant under the extendable G-action on (S^3, Σ_g). Since $|G| = 12(g-1)$, each handlebody orbifold H_i/G has the underlying space B^3 and singular set as indicated in Figure 2 (a). Then the quotient-orbifold S^3/G is S^3, the singular set is a 2-bridge link with the two standard unknotting tunnels of branching orders p and q added, with $p, q \leq 5$, as indicated in Figure 2(b); three of the strands of the 2-bridge link have branching order 2, the remaining one branching order 3, where σ is a braid on 3 strands, and these are the orbifolds $O(\sigma; p, q)$, see [Zi3] for details.

All spherical 3-orbifolds S^3/G with underlying space $|S^3/G| = S^3$ are listed in Tables 6, 7, 8 of Dunbar's paper [Du1] (pages 89-93): Each singular set is a graph with vertex valency at most 3, and each edge is labeled by an integer indicating the singular index of the edge, with the convention that each unlabeled edge has index 2. Each small box encoded by an integer k indicates two parallel arcs with k-half twists in the box, and each small box encoded by two integers m, n indicates a rational tangle given by (m, n) with a "strut" connecting the two arcs of the tangle and labeled by the largest common divisor of m and n.

Now we use Dunbar's list [Du1] to see which of the orbifolds in Figure 2 are spherical. There are two cases.

(1) If not both p and q are equal to 2 then there is a singular point which is not dihedral, so it is of type A_4 (tetrahedral), S_4 (octahedral) or A_5 (icosahedral). In Dunbar's list, these are only the non-fibered orbifolds on page 93 of [Du1]. By further checking which graph can meet a 2-sphere S^2 with four singular points of indices $(2, 2, 2, 3)$ so that each side of S^2 is a handlebody orbifold, we have only nine graphs left which are listed in Figure 3. From [Du2], we also know the fundamental groups of these nine orbifolds, and we indicate the groups and their orders under each graph. Here O denotes the orientation-preserving symmetry group of the octahedron, and J the orientation-preserving symmetry group of the icosahedron; all the symbols are from [Du2].

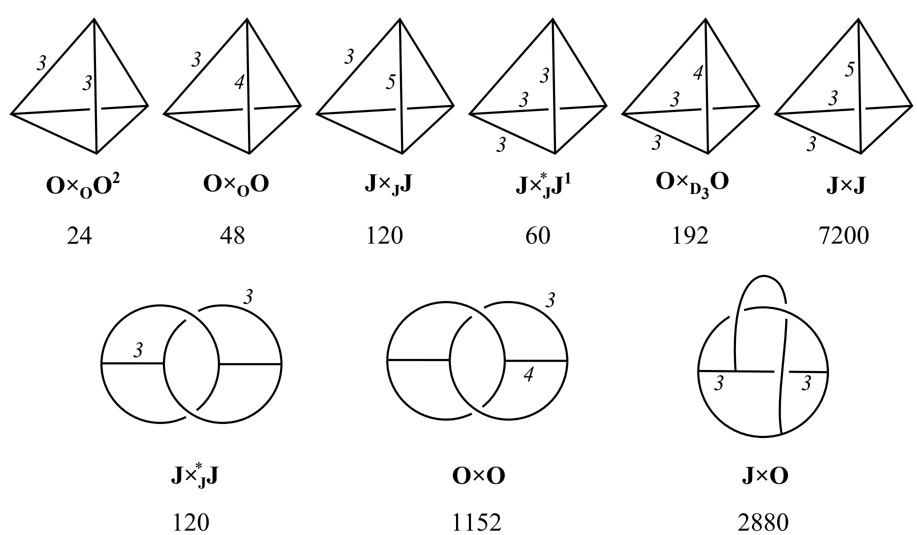

Figure 3

From the relation $|G| = 12(g-1)$, the cases one finds are $g = 3$, 5, 11, 6, 17, 601, again 11, 97 and 241.

(2) If both p and q are equal to 2, then an easy and more interesting way to find the cases is as follows. Take any 2-bridge link $L(\sigma)$ in S^3 and associate branching index 3 to each of its components, obtaining an orbifold $L_3(\sigma)$. It is shown in [MeZ] that such an orbifold $L_3(\sigma)$ has an orientation-preserving symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the quotient orbifold is exactly $O(\sigma; 2, 2)$, for the same 2-bridge knot defined by σ. So if $O(\sigma; 2, 2)$ is spherical, also $L_3(\sigma)$ is spherical and the singular set is just a 2-bridge link now. The spherical orbifolds whose singular set is just a link are listed on pages 89-92 of [Du1]. When we restrict to the two bridge links such that each component has index 3, then only five links are left which are listed in Figure 4, and all of them are very simple torus links.

Figure 4

Let $\tilde{L}_3(\sigma)$ be the 3-fold cyclic branched covering of $L_3(\sigma)$ so that $\tilde{L}_3(\sigma)$ has no singularity. Then we have the orbifold covering

$$\tilde{L}_3(\sigma) \to L_3(\sigma) \to O(\sigma; 2, 2)$$

of degree 12.

For all $L_3(\sigma)$ in Figure 4, $\tilde{L}_3(\sigma)$ are well-known spherical Seifert fiber spaces: The first is S^3, the second the lens space space $L(3, 1)$, the third is the quaternion manifold, the fourth is the 3-manifold with binary tetrahedral fundamental group, and the fifth is the Poincare homology 3-sphere (indeed these five 3-manifolds are exactly the k-fold cyclic branched covering over the trefoil knot, for $k = 1$, 2, 3, 4 and 5; see [Ro], pp 304-309 for a discussion of the cyclic branched coverings of the trefoil). The fundamental groups of these orbifolds $\tilde{L}_3(\sigma)$ have orders 1, 3, 8, 24 and 120, so for the corresponding orbifolds $O(\sigma; 2, 2)$ one has orders 12, 36, 96, 288 and 1440; applying again $|G| = 12(g-1)$ we obtain the genera 2, 4, 9, 25 and 121. $\square$

We end this section by a lemma which should be useful to study Question 1, and has also some applications in Section 4.

Lemma 2.2. *Suppose a group G acts on (M, F) where M is a 3-manifold and $F \subset M$ a surface, so we have the diagrams:*

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "p"'] & M \arrow[d, "p"] \\ F/G \arrow[r, "\hat{i}"] & M/G \end{tikzcd} \qquad \begin{tikzcd} \pi_1(F) \arrow[r, "i_*"] \arrow[d, "p_*"'] & \pi_1(M) \arrow[d, "p_*"] \\ \pi_1(F/G) \arrow[r, "\hat{i}_*"] & \pi_1(M/G) \end{tikzcd}$$

Suppose F/G is connected. Then F is connected if

$$\hat{i}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M)) = \pi_1(M/G).$$

Proof. Suppose F is not connected. Let $F_1 \subseteq F$ be a component of F and G_1 its stabilizer in G, that is $G_1 = \{h \in G | h(F_1) = F_1\}$. Then $F_1/G_1 = F/G$. Now $|\pi_1(M/G) : p_*(\pi_1(M))| = |G|$, and

$$\begin{aligned}
& |\hat{i}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M)) : p_*(\pi_1(M))| \\
= \; & |\hat{i}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M))/p_*(\pi_1(M))| \\
= \; & |\hat{i}_*(\pi_1(F/G))/\hat{i}_*(\pi_1(F/G)) \cap p_*(\pi_1(M))| \\
\leq \; & |\hat{i}_*(\pi_1(F/G)) : \hat{i}_* p_*(\pi_1(F_1))| \\
= \; & |\pi_1(F/G))/ker\hat{i}_* : p_*(\pi_1(F_1)) \cdot ker\hat{i}_*/ker\hat{i}_*| \\
= \; & |\pi_1(F_1/G_1)) : p_*(\pi_1(F_1)) \cdot ker\hat{i}_*| \\
\leq \; & |\pi_1(F_1/G_1)) : p_*(\pi_1(F_1))| \\
= \; & |G_1| < |G|.
\end{aligned}$$

Hence $\hat{i}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M)) \subsetneqq \pi_1(M/G)$. □

Remark In Lemma 2.2, if M is S^3, then F is connected if $\hat{i}_*$ is surjective.

3 Maximum orders of extendable abelian and cyclic groups

To get the maximum orders AE_g and CE_g, we first need some information about actions of abelian groups and cyclic groups on handlebodies which are contained in the following Theorem 3.1 and Theorem 3.2. Some facts in Theorem 3.1 and Theorem 3.2 have been either explicitly or implicitly stated, with or without proofs, in [MMZ], [MeZ]. For our later applications, we reorganize them into our statement and provide proofs.

We define two actions of a finite group G to be *equivalent* if the corresponding groups of homeomorphisms of V_g are conjugate (i.e., allowing isomorphisms of G).

Theorem 3.1. *The largest order of a finite abelian group G acting on the handlebody V_g of genus $g \geq 2$ is $2(g+1)$ if $g \neq 5$, and 16 if $g = 5$. The groups G which realize the maximum orders are $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ if $g \neq 5$, $(\mathbb{Z}_2)^4$ if $g = 5$, and in addition $(\mathbb{Z}_2)^3$ if $g = 3$. Moreover,*

(i) there is one equivalence class for each of the groups $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ and $(\mathbb{Z}_2)^4$ whereas there are three equivalence classes for the group $(\mathbb{Z}_2)^3$ acting on V_3;

(ii) no abelian group of order larger than 12 acts on V_5 except $(\mathbb{Z}_2)^4$.

Proof. Suppose that G is an abelian group as in Theorem 3.1. Then $g \geq 2$ implies $-\chi > 0$ by (2.2). Also, every vertex group of $(\Gamma, \mathcal{G})$ is either cyclic or isomorphic to the dihedral group $(\mathbb{Z}_2)^2$ of order 4, since these are the finite abelian subgroups of $SO(3)$; then the group of every edge which is not a loop is either trivial or $\mathbb{Z}_2$, and in the second case the two adjacent vertex group are $(\mathbb{Z}_2)^2$.

Note that, if $|G| \geq 2g-1$, then $-\chi = (g-1)/|G| \leq (g-1)/(2g-1) < 1/2$. We will assume that $-\chi < 1/2$ in the following and discuss all possibilities for $(\Gamma, \mathcal{G})$ and G. The discussion is divided into two cases:

(I) Suppose first that $(\Gamma, \mathcal{G})$ has no vertex group $(\mathbb{Z}_2)^2$; then all vertex groups are cyclic. Let

$$E = \{\text{edge } e \in \Gamma | G_e \text{ is non-trivial}\};$$

then all edges in E must be loops. Let $\Gamma_0 = \Gamma - E$; if we view Γ_0 as a usual graph, we denote it by $|\Gamma_0|$. It is easy to see $-\frac{1}{2} < \chi(\Gamma) \leq \chi(\Gamma_0) \leq \chi(|\Gamma_0|)$. However $\chi(|\Gamma_0|)$ is an integer, so $\chi(|\Gamma_0|) = 1$ or 0. Suppose Γ_0 has k non-trivial vertices; then $\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{k}{2}$. So if $\chi(|\Gamma_0|) = 0$, it is easy to see $\chi(\Gamma) \notin (-\frac{1}{2}, 0)$. So we must have $\chi(|\Gamma_0|) = 1$, which means $|\Gamma_0|$ must be a tree. Notice that every end (degree-one vertex) of Γ_0 must be non-trivial, hence the same reason as above shows that Γ_0 is equal to some $\Gamma(\mathbb{Z}_{n_1}, 1, \mathbb{Z}_{n_2})$ (consisting of one edge with trivial edge group, and two vertices). Furthermore, $(n_1, n_2) \in \{(3,5), (3,4), (3,3), (2,n)\}$, $n \geq 3$, and Γ must equal to Γ_0. So the only possibilities for Γ are the graphs of groups $\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)$, with $-\chi = (n-2)/2n$, and $\Gamma(\mathbb{Z}_3, 1, \mathbb{Z}_n)$, for $n = 3$, 4 or 5, with $-\chi = 1/3$, $5/12$ or $7/15$. The only finite abelian groups onto which the free product $\pi_1\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n) \cong \mathbb{Z}_2 * \mathbb{Z}_n$ surjects with torsionfree kernel are the groups $\mathbb{Z}_2 \times \mathbb{Z}_n$ and, if n is even, $\mathbb{Z}_n$. In the first case we have $g = n-1$ and $|G| = 2n = 2(g+1)$, so for all genera g the order $2(g+1)$ of the Theorem is achieved for the group $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$. In the three other cases, the possibilities for G are the groups $\mathbb{Z}_3$ and $(\mathbb{Z}_3)^2$, $\mathbb{Z}_{12}$ or $\mathbb{Z}_{15}$, and in each of these cases one has $|G| < 2(g+1)$.

(II) Suppose now that $(\Gamma, \mathcal{G})$ has some vertex group $(\mathbb{Z}_2)^2$. Let

$$E = \{\text{edge } e \in \Gamma | G_e \text{ is non-trivial, and the ends of } e \text{ are both cyclic groups}\};$$

also all edges in E must be loops. Setting $\Gamma_0 = \Gamma - E$, every non-trivial edge in Γ_0 must have both ends $(\mathbb{Z}_2)^2$. We now also have $-\frac{1}{2} < \chi(\Gamma) \leq \chi(\Gamma_0) \leq \chi(|\Gamma_0|)$. We explain the last inequality: suppose Γ_0 has l vertices of type $(\mathbb{Z}_2)^2$; then Γ_0 has no more than $\frac{3l}{2}$ non-trivial edges, so $\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{3}{4}l + \frac{1}{2} \cdot \frac{3l}{2} = \chi(|\Gamma_0|)$. We also have $\chi(|\Gamma_0|) = 1$ or 0. Now for a cyclic end, $\chi(\Gamma_0)$ must decrease at least by $-\frac{1}{2}$ compared with $\chi(|\Gamma_0|)$. For a $(\mathbb{Z}_2)^2$ end, Γ_0 has no more than $\frac{3l-2}{2}$ non-trivial edges, and this time $\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{3}{4}l + \frac{1}{2} \cdot \frac{3l-2}{2} = \chi(|\Gamma_0|) - \frac{1}{2}$, so it also decrease at least by $-\frac{1}{2}$. This argument show that Γ_0 has at most 2 ends.

(1) If $\chi(|\Gamma_0|) = 0$, Γ_0 can even have no ends at all, so it is a loop divided by some $(\mathbb{Z}_2)^2$ vertices. The only possibility for $(\Gamma, \mathcal{G})$ is a graph of groups with exactly one vertex and one edge (a loop), with $\chi = -1/4$; however, since the HNN-generator of $\pi_1(\Gamma, \mathcal{G})$ corresponding to the loop has to conjugate a subgroup $\mathbb{Z}_2$ of the vertex group $(\mathbb{Z}_2)^2$ into a different subgroup $\mathbb{Z}_2$ (see [MMZ] or [Zi3]), its fundamental group does not surject onto an abelian group and this case does not occur.

(2) If $\chi(|\Gamma_0|) = 1$, Γ_0 is a segment. The segment may have inner vertices, but every inner vertex must be $(\mathbb{Z}_2)^2$. For every such inner vertex, Γ_0 has no more than $\frac{3l-1}{2}$ non-trivial edges, and this time $\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{3}{4}l + \frac{1}{2} \cdot \frac{3l-1}{2} = \chi(|\Gamma_0|) - \frac{1}{4}$. So there is at most one inner vertex. If there is no inner vertex, it is easy to see that $(\Gamma, \mathcal{G})$ is equal to $\Gamma((\mathbb{Z}_2)^2, 1, \mathbb{Z}_2)$, with $-\chi = 1/4$, or to $\Gamma((\mathbb{Z}_2)^2, 1, \mathbb{Z}_3)$, with $-\chi = 5/12$; the possibilities for G are the groups $(\mathbb{Z}_2)^2$ and $(\mathbb{Z}_2)^3$ in the first case, and $\mathbb{Z}_2 \times \mathbb{Z}_6$ in the second one, and only

for the group $(\mathbb{Z}_2)^3$, with $g = 3$, the bound $|G| = 2(g+1)$ of the Theorem is obtained. The last case is the graph of groups $\Gamma((\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2)$, with two edges and three vertices and $-\chi = 1/4$, whose fundamental group surjects onto $(\mathbb{Z}_2)^n$ for $n = 2$, 3 and 4; the group $(\mathbb{Z}_2)^4$ realizes the maximum order 16 for $g = 5$ of the Theorem, whereas the group $(\mathbb{Z}_2)^3$ realizes again the maximum order $8 = 2(g+1)$ for $g = 3$

There is only one finite-injective surjective map from $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ or from $\Gamma((\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2)$ to an abelian group. There are two finite-injective surjective map from $\Gamma((\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2)$ to $(\mathbb{Z}_2)^3$ (either all three vertex groups $(\mathbb{Z}_2)^2$ are mapped to different subgroups of $(\mathbb{Z}_2)^3$, or two vertex groups are mapped to the same subgroup), and one finite-injective surjective map from $\Gamma((\mathbb{Z}_2)^2, 1, \mathbb{Z}_2)$ to $(\mathbb{Z}_2)^3$. These show the result in (i). For (ii), an abelian group of order 13, 14 or 15 must be a cyclic group, and the above orbifolds can not finite-injectively surject to such a cyclic group when $g = 5$.

This completes the proof of Theorem 3.1. $\square$

For cyclic groups, the proof of Theorem 3.1 implies also the following:

Theorem 3.2. *Let G be a finite cyclic group acting on a handlebody of genus $g \geq 2$. If G has order at least $2g - 2$ then G is one the following groups:*

(1) $\mathbb{Z}_{2g+2}$ if g is even, associated to a surjection $\mathbb{Z}_2 * \mathbb{Z}_{g+1} \to \mathbb{Z}_{2(g+1)}$;

(2) $\mathbb{Z}_{2g}$ for all g, associated to a surjection $\mathbb{Z}_2 * \mathbb{Z}_{2g} \to \mathbb{Z}_{2g}$ and, for $g = 6$, also to $\mathbb{Z}_3 * \mathbb{Z}_4 \to \mathbb{Z}_{12}$;

(3) $\mathbb{Z}_{2g-1}$ for $g = 2$ and 8, associated to surjections $\mathbb{Z}_3 * \mathbb{Z}_3 \to \mathbb{Z}_3$ and $\mathbb{Z}_3 * \mathbb{Z}_5 \to \mathbb{Z}_{15}$;

(4) $\mathbb{Z}_{2g-2}$ for all g; for each g the graphs of groups are $\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)$ with an additional loop with edge group $\mathbb{Z}_n$ attached to the vertex of type $\mathbb{Z}_n$, for each $n \geq 1$ which divides $2g - 2$; in addition, for $g = 3$ and 2 there are actions associated to $\mathbb{Z}_4 * \mathbb{Z}_4 \to \mathbb{Z}_4$ and $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2$.

Proof. The cases $|G| \geq 2g - 1$ are covered by the proof of Theorem 3.1. For $|G| = 2g-2$ one has to consider in addition graphs of groups with $-\chi = 1/2$. If Γ_0 has at least three non-trivial vertices, it gives the case $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2$. Or $\Gamma_0 = \Gamma(\mathbb{Z}_{n_1}, 1, \mathbb{Z}_{n_2})$ and there are two more cases: (n_1, n_2)=$(4, 4)$, or Γ is obtained from Γ_0 by adding a non-trivial loop as stated in (4). $\square$

Theorem 3.3. $AE_g = 2g + 2$.

Proof. Suppose an abelian group G acts on Σ_g which is extendable over S^3 for some embedding $e : \Sigma_g \hookrightarrow S^3$. Then the action of G extends to each 3-manifold of $S^3 \setminus \Sigma_g$. According to [RZ, Theorem 2], for each abelian group G, the G-action on Σ_g extends to a compact 3-manifold M with $\partial M = \Sigma_g$ if and only if the G-action on Σ_g extends to a handlebody V_g with $\partial V_g = \Sigma_g$. Therefore we have $AE_g \leq AH_g$.

It is a general fact of Smith fixed point theory that the finite 2-group $(\mathbb{Z}_2)^{(n+1)}$ does not act orientation-preservingly on a *mod* 2 homology n-sphere, see [Sm]. So a $(\mathbb{Z}_2)^4$-action on F_5 does not extend to an action on S^3. (Alternatively, by the confirmation of the geometrization conjecture one

can use the fact that every finite group acting orientation-preservingly on S^3 can be conjugated into $SO(4)$, and $SO(4)$ contains no subgroup isomorphic to $(\mathbb{Z}_2)^4$.) Now applying Theorem 3.1, we have indeed $AE_g \leq 2(g+1)$ for each $g > 1$.

By Example 4.1, for every $g > 1$ there is an abelian group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ which acts on Σ_g, and this action extends to a G-action on S^3 for the unknotted embedding of $\Sigma_g \subset S^3$. Hence $AE_g^o \geq 2(g+1)$.

Then

$$2(g+1) \leq AE_g^o \leq AE_g \leq 2(g+1).$$

So we have $AE_g = AE_g^o = 2(g+1)$. and Theorem 3.3 is proved. $\square$

The following fact proved without using Smith theory is of independent interest.

Lemma 3.4. *Some order 2 element of the $(\mathbb{Z}_2)^4$ action on V_5 is not extendable.*

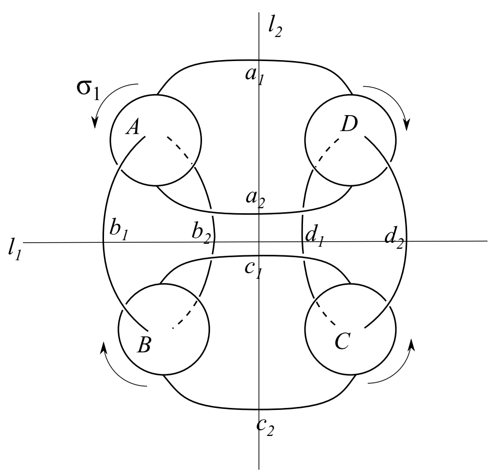

Figure 5

Proof. We first describe a geometric model of the action of $(\mathbb{Z}_2)^4$ on V_5.

As in Figure 5, we view V_5 as four 3-balls $\{A, B, C, D\}$ with 8 handles $\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$ attached; here, for simplicity, we draw each handle as an arc, and the four attaching disks on each ball are at front, back, top and bottom respectively.

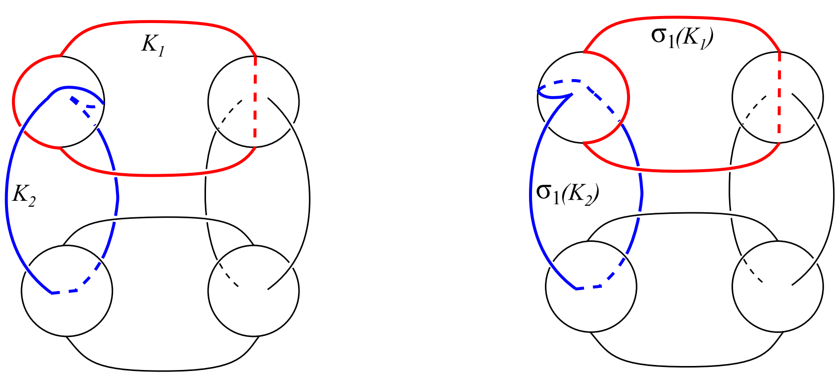

Figure 6

We will describe the four generators $\{\sigma_1, \sigma_2, \rho_1, \rho_2\}$ of the action of $G \cong (\mathbb{Z}_2)^4$: on each ball σ_1 is a π-rotation about the axis through the front and back points along the arrows shown as in Figure 6, which interchanges a_1 and a_2 (resp. c_1 and c_2) but keeps each b_i and d_i, $j = 1, 2$. Similarly $\sigma_2 \in G$ is a π-rotation about the axis through the top and bottom points on each ball, which interchanges b_1 and b_2, (resp. d_1 and d_2), but keeps each a_i and c_i, $j = 1, 2$. The generator ρ_i is a π-rotation of the whole handlebody about the axis l_i, $i = 1, 2$. One can check that $\{\sigma_1, \sigma_2, \rho_1, \rho_2\}$ generates the abelian group G. Note that ρ_1 and ρ_2 are extendable.

For any embedding $V_5 \hookrightarrow S^3$, we construct a link $\{K_1, K_2\}$ in the handlebody V_5 as in Figure 6 where the image $\sigma_1\{K_1, K_2\}$ is shown on the right hand side. The linking numbers locally differ by 1 in the upper-left ball. So σ_1 is not extendable. $\square$

Theorem 3.5. *$CE_g = 2g + 2$ if g is even, and $2g - 2$ if g is odd.*

Proof. In Example 4.1 we shall describe a cyclic group action of order $2g+2$ on Σ_g which extends over S^3 for each even $g > 1$, and in Example 4.2 a cyclic group action of order $2g - 2$ on Σ_g which extends over S^3 for each odd $g > 1$. Hence

$$CE_g \geq 2g + 2 \text{ for even } g > 1, \;\; CE_g \geq 2g - 2 \text{ for odd } g > 1. \qquad (3.3)$$

Suppose the G action on Σ_g is extendable. Still applying [RZ, Theorem 2], we have that the G action on Σ_g extends to $(V_g, \partial V_g = \Sigma_g)$.

By Theorem 3.2 (1) for each even g we have $AH_g = 2g+2$. By Theorem 3.2 (2) (3) for each odd g a cyclic group G of order $|G| > 2g - 2$ acting on V_g must be $\mathbb{Z}_{2g}$, is associated to surjection $\mathbb{Z}_2 * \mathbb{Z}_{2g} \to \mathbb{Z}_{2g}$.

Claim: *The $\mathbb{Z}_{2g}$ action on $\partial V_g = \Sigma_g$ which is the restriction of the $\mathbb{Z}_{2g}$ acts on V_g is not extendable.*

With this Claim we have

$$CH_g \leq 2g + 2 \text{ for even } g > 1, \;\; CH_g \leq 2g - 2 \text{ for odd } g > 1. \qquad (3.4)$$

Combining (3.3) and (3.4), Theorem 3.5 is proved.

Proof of the Claim. We must have a close look on the $\mathbb{Z}_{2g}$ action on V_g. By the discussion made in [MMZ], [Zi3], the handlebody orbifold $X = V_g/\mathbb{Z}_{2g}$ must consist of two 3-balls with singular arcs of indices 2 and $2g$, respectively, connected by a regular 1-handle as shown in the left hand side of Figure 7. Now the pre-image of the 3-ball with singular arc of index $2g$ is just an ordinary 3-ball B^3 in V_g, the $\mathbb{Z}_{2g}$-action on it is a $\frac{\pi}{g}$-rotation, and the pre-image of the remaining part of the handlebody orbifold X in V_g consists just of g 1-handles attached to opposite $\mathbb{Z}_{2g}$-equivariant disks on B^3. The right hand side of Figure 7 is the case of V_3.

Hence the $\mathbb{Z}_{2g}$ action on $\Sigma_g = \partial V_g = S^2_* \cup \{N_1, ..., N_g\}$ is obtained from the 2-sphere S^2_* with $2g$ punctures by attaching g tubes $N_1, ..., N_g$ along g pairs of opposite punctures.

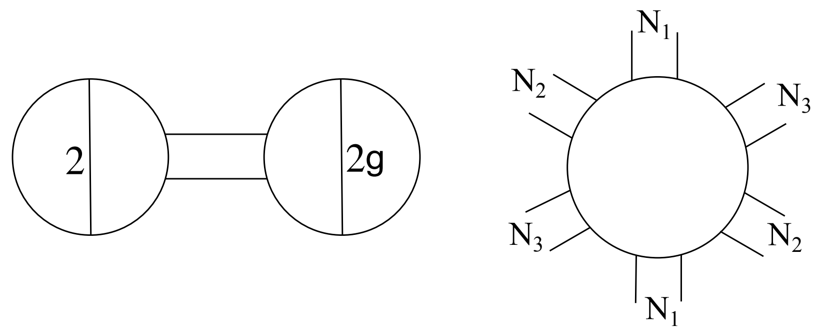

Figure 7

Now we are going to give two different proofs that this $\mathbb{Z}_{2g}$ action on Σ_g is not extendable.

Proof 1. The first one invokes Smith theory.

Suppose the $\mathbb{Z}_{2g}$ action on Σ_g extends an action on (S^3, Σ_g) for some embedding $\Sigma_g \to S^3$, and let σ be a generator of this extension. Since $\sigma|S^2_*$ is a rotation of order $2g$ with two fixed points, the fixed point set of σ is not empty. By Smith theory the fixed point set of the group $< \sigma >$ acting on S^3 must be a circle C. It follows that the (singular or branching) index of $C \cap S^2_*$ must be $2g$, therefore the index of the whole circle C must be $2g$, that is to say the whole C is the fixed point set σ. On the other hand, σ^g is a π-rotation on the tube N_1 which has two fixed points x, y, therefore $x \in C$ which implies that σ has fixed points on N_1. Since $g > 1$, σ sends the whole N_1 to N_2 which gives a contradiction.

Proof 2. The second one is elementary and using linking number only.

Choose an arc γ on the boundary of the orbifold X, as showed in Figure 8. The pre-image of γ consists of g arcs γ_i, $i = 1, \ldots, g$ on the surface Σ_g, equivariant under the action of G. Let D denote the upper hemisphere of the $2g$-punctured sphere S^2_* described above. Then the boundary of γ_i divides ∂D into $2g$ arcs denoted by α_i and β_i such that the $\frac{\pi}{g}$-rotation maps α_i to β_i; see the right hand side of Figure 8.

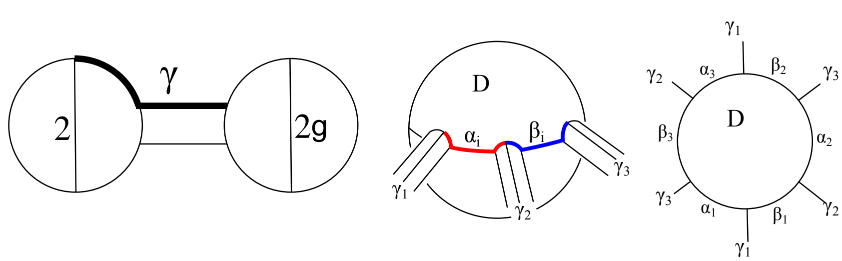

Figure 8

Now consider the embedding $\Sigma_g \hookrightarrow S^3$, and let

$$K_1 = \gamma_i \bigcup \alpha_i, \quad K_2 = \gamma_i \bigcup \beta_i$$

which are knots in S^3, see Figure 9.

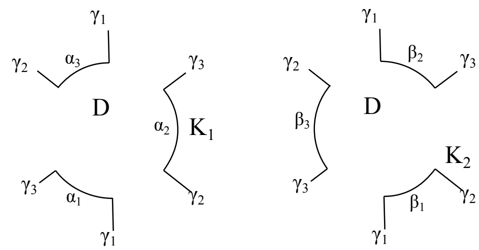

Figure 9

Suppose the G-action extends to a G-action on S^3, and let $\sigma \in G$ be the generater of G such that the restriction of σ to D is the $\frac{\pi}{g}$-rotation. Then we have $\sigma(K_1) = K_2$. The fixed point set of the periodic map σ is not empty (it has a fixed point x on D). Denote by K_0 the circle component of the fixed point set of σ containing x. Now $\sigma\{K_0, K_1\} = \{K_0, K_2\}$.

Let us compute the mod 2 linking numbers $lk_2(K_0, K_1)$ and $lk_2(K_0, K_2)$ to reach a contradiction. Choose a standard embedded S^2 in S^3, which contains the disk D. Project each knot K_i and K_0 to this S^2; we use this projected diagram to compute the linking numbers $lk_2(K_0, K_i)$: for each crossing, if the arc K_0 goes over the arc of K_i, then this crossing contributes 1 to $lk_2(K_0, K_i)$, and if the arc K_0 goes under the arc of K_i then this crossing contributes 0 to $lk_2(K_0, K_i)$ (this is just an application of the general linking number method, see [Ro] for details).

We notice that the only differences between $lk_2(K_0, K_1)$ and $lk_2(K_0, K_2)$ are coming from the crossings of K_0 with ∂D. There are three cases, shown in Figure 10.

Case 1. Both ends of the arc in the disk go over ∂D. If the two ends are both over α_i or both over β_i, then it contribute 0 to both $lk_2(K_0, K_1)$ and $lk_2(K_0, K_2)$. If one end goes over α_i and the other goes over β_i then this arc contributes 1 to both $lk(K_0, K_1)_2$ and $lk_2(K_0, K_2)$.

Case 2. Both ends of the arc in the disk go under ∂D. In this case the arc contributes 0 to both $lk_2(K_0, K_1)$ and $lk_2(K_0, K_2)$.

Case 3. One end of the arc in the disk goes over ∂D, and the other end goes under ∂D. In this case, the arc of K_0 must intersect D in its interior, and that means it goes through the only fixed point in D. So there is exactly one such arc. If the "over end" is over α_i, then this arc contributes 1 to $lk_2(K_0, K_1)$ but 0 to $lk_2(K_0, K_2)$. If the "over end" is over β_i then this arc contributes 0 to $lk_2(K_0, K_1)$ but 1 to $lk_2(K_0, K_2)$. Anyway, it is not the same for the two linking numbers.

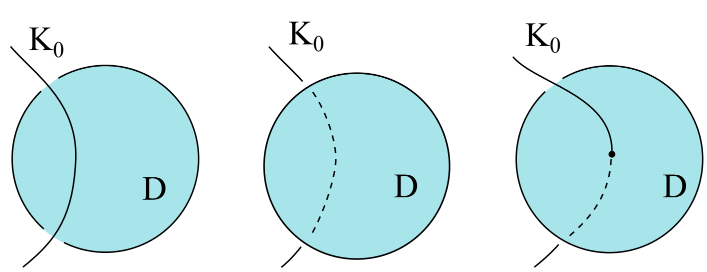

Figure 10

From the above, we must have that

$$lk_2(K_0, K_1) \neq lk_2(K_0, K_2).$$

Since σ is an automorphism of S^3 which send $\{K_0, K_1\}$ to $\{K_0, K_2\}$, this is a contradiction. So for $|G| = 2g$ the G-action on Σ_g does not extend to a G-action on S^3. □

4 Intuitive view of large symmetries of (S^3, Σ_g)

In this section we will present some extendable group actions on surfaces with "large" symmetry. Also if the existence of such examples often can be derived from the powerful orbifold theory, we would like to see how these symmetries stay in the symmetry of our 3-sphere in a more direct and intuitive way, as we mentioned in the end of the introduction.

Remark: In the ten examples below, the first seven are constructed before applying orbifold theory to get the result in Section 2 and 3, and the last three examples was constructed after we knowing results in Sections 2 and 3. The constructions of first nine examples use no information from orbifold theory.

Example 4.1 For every $g > 1$, there is an abelian group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ which acts on Σ_g such that the action extends to a G-action on S^3, for the standard embedding of $\Sigma_g \subset S^3$. When g is even, we get a cyclic group action of order $2g + 2$ on Σ_g which extends over S^3.

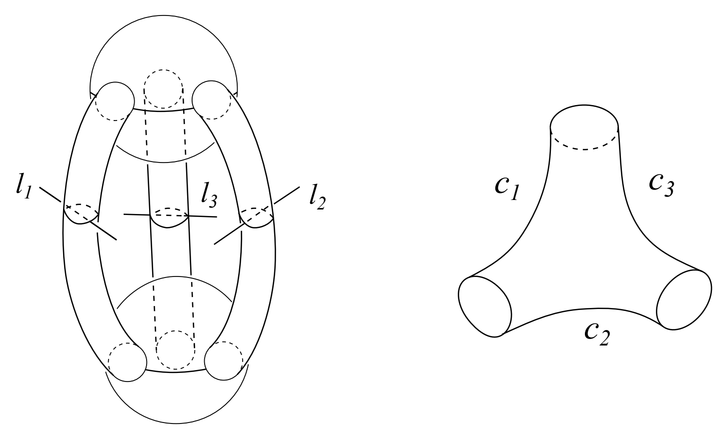

Figure 11

We embed V_g in S^3 as two 3-balls together with $g + 1$ handles attached to the equator, see Figure 11 for $g = 2$. One can easily see a $\mathbb{Z}_{g+1}$-action by a rotation on S^3 which keeps V_g invariant. We construct a $\mathbb{Z}_2$ involution on V_g as follows: The involution restricted to each handle is a π-rotation about each line l_i drawn in Figure 11, and the involution maps each of the two 3-balls to the other one, without any rotation or reflection. One can carefully check the attaching disks to show this is well defined; moreover the involution and the rotation commutate and give an action of $\mathbb{Z}_{g+1} \times \mathbb{Z}_2$ on (S^3, V_g). When g is even, the composition of the involution and the $\frac{2\pi}{g+1}$-rotation is an order $2g + 2$ map.

Another simple and useful way to observe this action is just to think about a pair of pants, on the right hand side of Figure 11. Its neighborhood in S^3 is a genus 2 handlebody. This pair of pants consists of two pieces of cloth sewn together along three lines c_i. So this is just the same as on the left hand side, letting each piece of cloth correspond to a 3-ball and each c_i to a handle. Now the rotation on the pair of pants is quite obvious and is extendable. The involution just changes the positions of the two pieces of cloth, keeping c_i fixed. This makes the pants inner to outer, and because you can really do this practically, it extends to an involution of S^3.

Example 4.2 Now we construct an example of a $\mathbb{Z}_{2g-2}$ action on V_g for g odd. Let us consider S^3 as the unit sphere in $\mathbb{C}^2$,

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

There is a solid torus $T \in S^3$,

$$T = \{(z_1, z_2) \in S^3 \mid |z_1| \leq \frac{\sqrt{2}}{2}\}.$$

We choose $g-1$ pairs of points a_k and b_k, $k = 1, 2, \ldots, g-1$,

$$a_k = (\frac{\sqrt{2}}{2}e^{\frac{2k\pi i}{g-1}}, \frac{\sqrt{2}}{2}e^{\frac{k\pi}{g-1}}),$$

$$b_k = (\frac{\sqrt{2}}{2}e^{\frac{2k\pi i}{g-1}}, \frac{\sqrt{2}}{2}e^{\frac{(k+g-1)\pi i}{g-1}}).$$

In the disk

$$D_k = \{(re^{\frac{2k\pi i}{g-1}}, z_2) \in S^3 \mid r \geq \frac{\sqrt{2}}{2}\},$$

there is a unique diameter γ_k connecting a_k and b_k. Let N_k be a neighborhood of γ_k in S^3; then the solid torus T together with the $(g-1)$ handles N_k gives an embedding of V_g into S^3.

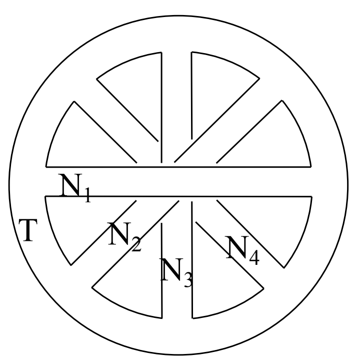

Figure 12

Now the $(2g-2)$ order action on S^3 can be described like this:

$$\sigma : S^3 \to S^3$$

$$(z_1, z_2) \mapsto (z_1 e^{\frac{2k\pi i}{g-1}}, z_2 e^{\frac{k\pi i}{g-1}}).$$

This action keeps T invariant and sends each N_k to $N_{k+1}(mod(g-1))$. A rough picture is showed in Figure 12 for $g = 5$.

Example 4.3 For every $g > 1$, we will construct a group G of order $4(g+1)$ which acts on $S^3 = V_g \bigcup V_g'$. We consider V_g as the neighborhood of a sphere S^2 with $g+1$ punctured holes. We choose the holes all on the equator, centered at the vertices of a regular $g+1$-polygon. There is a dihedral group D_{g+1} acting on S^2 which keeps the holes invariant (as a set). And there is also a $\mathbb{Z}_2$ action changing the inner and outer of S^2, as described in Example 4.1. So there is a $D_{g+1} \times \mathbb{Z}_2$ action on V_g. This group has order $4(g+1)$.

Example 4.4 For every square number $g > 1$, we construct a group G of order $4(\sqrt{g}+1)^2$ which acts on $S^3 = V_g \bigcup V_g'$ (notice that this is greater than $4(g+1)$).

With $g = k^2$, the group G is a semidirect product

$$(\mathbb{Z}_{k+1} \times \mathbb{Z}_{k+1}) \rtimes_\varphi (\mathbb{Z}_2 \times \mathbb{Z}_2).$$

Writing $\mathbb{Z}_{k+1} \times \mathbb{Z}_{k+1} = \langle x, y | xy = yx, x^{k+1} = y^{k+1} = 1 \rangle$, $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle s, t | st = ts, s^2 = t^2 = 1 \rangle$, the semidirect product is given by

$$\varphi : sxs^{-1} = y, \quad sys^{-1} = x, \quad txt^{-1} = x^{-1}, \quad tyt^{-1} = y^{-1}.$$

Consider S^3 as the unit sphere in $\mathbb{C}^2$

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

and let

$$a_j = (e^{\frac{2j\pi i}{k+1}}, 0), \qquad b_j = (0, e^{\frac{2j\pi i}{k+1}}), \qquad j = 0, 1, \ldots, k.$$

Then the G-action on S^3 is given by:

$$x : (z_1, z_2) \mapsto (e^{\frac{2\pi i}{k+1}} z_1, z_2),$$
$$y : (z_1, z_2) \mapsto (z_1, e^{\frac{2\pi i}{k+1}} z_2),$$
$$s : (z_1, z_2) \mapsto (z_2, z_1),$$
$$t : (z_1, z_2) \mapsto (\bar{z_1}, \bar{z_2}).$$

It is easy to check this is a faithful orientation-preserving action.

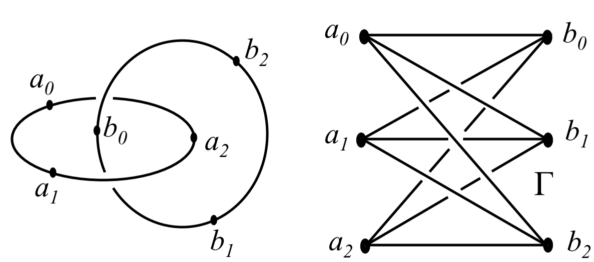

Figure 13

Notice that this G-action keeps the set $\{a_i, b_j\}, i, j = 0, 1, \ldots, k$, invariant. If we join each a_i and b_j by a geodesic in S^3, we get a two-parted graph $\Gamma \in S^3$ with $2k+2$ vertices and $(k+1)^2$ edges. Hence $\chi(\Gamma) = -k^2 + 1$. A neighborhood of Γ in S^3 is a handlebody of genus $k^2 = g$. Since the G-action

maps Γ to itself, it induces an action on V_g. Figure 13 gives the picture for $g = 4$.

For $g = 4$, this example gives a group action of order 36, which is maximal.

Example 4.5 We view $S^3 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 1\}$ and $S^2 = \{(x, y, z, 0) \mid x^2 + y^2 + z^2 = 1\} \subset S^2$. For each finite group $G \leq O(3)$ which acts on $S^2 \subset R^3$, we define $\widetilde{G} \leq SO(4)$ acts on S^3 which keeps the S^2 invariant as below: For each $\sigma \in G$, define $\widetilde{\sigma} : S^3 \to S^3$ by

$$\widetilde{\sigma}(x, y, z, w) \mapsto \begin{cases} (\sigma(x, y, z), w) & \sigma \text{is orientation-preserving;} \\ (\sigma(x, y, z), -w) & \sigma \text{is orientation-reversing.} \end{cases}$$

Now if we choose G to be the symmetry group of a tetrahedron with vertices on this S^2, then $G \cong S_4$. Let $\{v_1, v_2, v_3, v_4\}$ be the vertices of this tetrahedron, then G and $\widetilde{G}$ keeps this vertices set invariant. If we choose 4 holes puncturing this S^2 at the positions of $\{v_1, v_2, v_3, v_4\}$, we get a 4-punctured sphere X, its neighborhood in S^3 is a handlebody $V_3 = N(X)$. And $\widetilde{G}$ act on (V_3, S^3). Note that $\widetilde{G} \cong G \cong S_4$, $|\widetilde{G}| = 24$, this gives an example for $g = 3$ and $|\widetilde{G}| = 24 = 12(g - 1)$.

Similarly we can choose G to be the symmetry group of a cube or a dodecahedron, we get $\widetilde{G} \cong S_4 \times \mathbb{Z}_2$ for $g = 5$ and $\widetilde{G} \cong A_5 \times \mathbb{Z}_2$ for $g = 11$. They all satisfies $|\widetilde{G}| = 12(g - 1)$.

Example 4.6 The quotients S^3/I^* of S^3 by the binary icosahedral group is the famous Poincaré homology 3-sphere which is also obtained by identifying pairs of faces of the dodecahedron. The covering $p : S^3 \to S^3/I^*$ provides a tessellation of S^3 by dodecahedra. Since the deck group is of order 120 and the symmetry group of the dodecahedron is 60, it is easy to see there is a group G of order $120 \times 60 = 7200$ acting on this tessellation, and in particular acting on the boundary surface Σ_g of a regular neighborhood of the 1-skeleton of this tessellation. Let v_i be the number of the i-dimensional cells of this tessellation; then $v_0 - v_1 + v_2 - v_3 = \chi(S^3) = 0$. Clearly $v_3 = 120$ and $v_2 = 120 \times 12/2 = 720$. So $v_1 - v_0 = v_2 - v_3 = 600$, and therefore $g = v_1 - v_0 + 1 = 601$ and $|G| = 7200 = 12(g - 1)$.

This example can be also considered as the orientation preserving symmetry group of the regular 120-cell in 4-space, in the spirit of the next example.

Example 4.7 Let Δ be the 4-dimensional regular Euclidean simplex and Θ be the 4-dimensional Euclidean cube centered at the origin of E^4 and inscribed in the unit sphere S^3. The radical projection of their boundaries to S^3 gives two regular tessellations of S^3, still denoted as $\partial\Delta$ and $\partial\Theta$ respectively.

Using the notions defined in Example 4.6, for $\partial\Delta$ we have $v_0 = v_3 = 5$, $v_1 = v_2 = 10$. Since $10 - 5 + 1 = 6$, the boundary surface of the regular neighborhood of the 1-skeleton of this tessellation is Σ_6. Each 3-dimensional face is a tetrahedron which has a symmetry of order 12, and then it follows that a group G of $60 = 12 \times 5 = 12(g - 1)$ acts on this tessellation, and in particular acts on Σ_6.

For $\partial\Theta$, since Θ is the product of the 3-dimensional cube with an inteval, it derived easily that we have $v_0 = 16$, $v_1 = 32$, $v_2 = 24$ and $v_3 = 8$. Since $32 - 16 + 1 = 17$, the boundary surface of the regular neighborhood of the 1-skeleton of this tessellation is Σ_{17}. Each 3-dimensional face is a 3-dimensional cube which has a symmetry of order 24, and it follows that a group G of order $24 \times 8 = 192 = 12(g-1)$ acts on this tessellation, and in particular on Σ_{17}.

So in this example we get actions of groups on (S^3, Σ_g) of maximaum order $12(g-1)$, for $g = 6$ and 17.

Example 4.8 Let $M' = P \times S^1$, where P is the oriented pair of pants, with the induced orientation on $\partial P = \{c_1, c_2, c_3\}$, and S^1 is oriented and represented by a curve h. Now attach 3 solid tori N_i along the first three boundary tori of M' so that the meridian of N_i is identified with a curve of slope $l_i = 2c_i + h$, $i = 1, 2, 3$. Denote the resulting manifold by M, which has the following properties:

(1) Recall that the full symmetry group of the pair of pants P is $G_P = D_3 \times \mathbb{Z}_2$, where the D_3 action on P is orientation preserving, and the $\mathbb{Z}_2$-action, which exchanges the inner and outer of P as described in Example 4.3, is orientation revering. We extend the $G_P = D_3 \times \mathbb{Z}_2$ action on P to $P \times S^1$ in an orientation preserving way by matching D_3 with the identity of S^1 and $\mathbb{Z}_2$ with an orientation reversing involution (reflection) on S^1. The latter extends over M since it preserves the set of attaching slopes.

(2) The map $\pi_1(P) \to \pi_1(M)$ is a surjection, therefore the map $\pi_1(\Sigma_2) \to \pi_1(M)$ is a surjection where Σ_2 is the boundary of the regular neighborhood of P in M; here all maps are induced by inclusions, and we use the fact that each attaching curve goes over the S^1 direction only once.

(3) M is a spherical 3-manifold with $\pi_1(M) = \mathbb{Z}_3 \times D_8^*$, where D_8^* is the quaternion group of order 8, see [Or], in particular $|\pi_1(M)| = 24$.

Now consider the covering $p : S^3 \to M$. By (2) and Lemma 2.2 the preimage of Σ_2 is connected, therefore it is the surface Σ_{25} by (3) which is invariant under the group of order $24 \times 12 = 12(g-1)$.

Example 4.9 We perform -1 surgery on each component of the link given in Figure 14. considering the Wirtinger presentation, the fundamental group of the resulting manifold is $< x, y, z | x = yz, y = zx, z = xy >$ which is isomorphic to the quaternion group Q by the map induced by $x \mapsto i, y \mapsto j, z \mapsto k$. The resulting manifold is the quaternion manifold S^3/Q. Consider a point in front of the paper and a point behind. Through each of the three components of the link choose a string connecting these two points. We get a θ-graph. The boundary of a neighborhood of the θ-graph is the surface Σ_2. As described before, there is a group of order 12 acting on S^3 which leaves this surface invariant. This action leaves also the link invariant and extends to the surgered solid tori. Lifting to S^3 we have a group action of order 96. Since $\pi_1(\theta)$ is generated by $x^{-1}z$ and $x^{-1}y$, it is easy to see the homomorphism $\pi_1(\theta) \to \pi_1(S^3/Q)$ is surjective. By Lemma 2.2, the lift of Σ_2 is connected and hence is Σ_9. Hence we get an extendable group action on Σ_9, of order $12(9-1) = 96$.

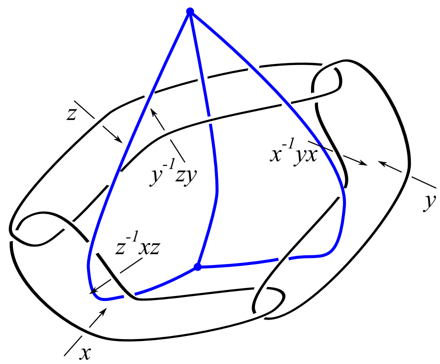

Figure 14

Similarly, performing $+1$ surgery on a trefoil indicated in Figure 15 we obtain the Poincaré homology 3-sphere P. Letting $u = yx$, the fundamental group has a presentation $\pi_1(P) =< u, x | u^3 = x^5 = (xu)^2 >$. It is isomorphic to the binary icosahedral group I^* by the map induced by $u \mapsto (1 + \sigma i + \delta j)/2, x \mapsto (\sigma + i - \delta k)/2$, where $\sigma = (\sqrt{5}+1)/2, \delta = (\sqrt{5}-1)/2$. We can similarly construct a θ-graph, and $\pi_1(\theta)$ is generated by $x^{-1}ux^{-1} \mapsto (\sigma + \delta j + k)/2$ and $ux^{-2} \mapsto (\sigma - \delta j + k)/2$. One can verify that $\pi_1(\theta) \to \pi_1(P)$ is surjective. Then we lift the neighborhood boundary surface of the θ-graph and the group action as before and obtain an extendable action on Σ_{121} of order $120 \times 12 = 1440 = 12(g-1)$.

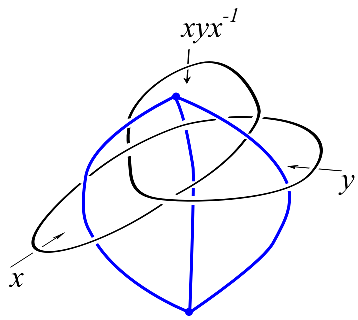

Figure 15

Example 4.10. Considering the 2:1 map $SO(4) \to SO(3) \times SO(3)$, the preimage of $O \times O$, denoted by $\mathbf{O} \times \mathbf{O}$, acts on S^3 (O denotes the octahedral group, of order 24). It has order $24 \times 24 \times 2 = 1152 = 12(97-1)$ and the pre-fundamental domain is a truncated cube [Du2]. If two such domains are adjacent via an octagon, we draw an edge between the centers of them. Then we get a graph, and the boundary of the regular neighborhood of this graph is a surface Σ_{97} with an action of $\mathbf{O} \times \mathbf{O}$ which is obviously extendable.

The preimage of $O \times J$, denoted by $\mathbf{O} \times \mathbf{J}$, acts on S^3. It has order $24 \times 60 \times 2 = 2880 = 12(241-1)$ and the pre-fundamental domain is a 'twice truncated tetrahedron'(or small tetrahedron in Dunbar's paper). If two such domains are adjacent via a dodecagon, we draw an edge between their centers. Then we get a graph, and the boundary of a regular neighborhood of this graph is a surface Σ_{241} with an extendable action of $\mathbf{O} \times \mathbf{J}$.

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